Relative Trisections of Smooth 4–manifolds With Boundary

by

Nickolas Andres Castro

(Under the Direction of David T. Gay)

Abstract

We show that every smooth, compact, connected, oriented 4–manifold with non-empty (connected or disconnected) boundary can be decomposed into three diffeomorphic 4–dimensional 1–handlebodies, $\natural^k S^1 \times B^3$, for some k. The pairwise intersections are compression bodies diffeomorphic to $\natural^k S^1 \times D^2$ and the triple intersection is a surface with boundary. Such a decomposition is called a *relative trisection.* Additionally, we define a stabilization technique for relative trisections which provides a more general uniqueness statement for relative trisections than that provided by Gay and Kirby. We also show that relatively trisected 4–manifolds can be glued together along diffeomorphic boundary components to induce a trisection of the resulting 4–manifold.

Index words: Trisections, Heegaard Splittings, Open book decomposition, Lefschetz fibration, Smooth 4–manifolds

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Chapter 1

Introduction

A trisection of a smooth, compact, connected, oriented 4–manifold is a decomposition $X = X_1 \cup X_2 \cup X_3$ into three diffeomorphic 4–dimensional 1–handlebodies ($X_i \cong \natural^k S^1 \times B^3$) with certain nice intersection properties. Trisections are the natural analog of Heegaard splittings of 3–manifolds, and there are striking similarities between the two theories. This opens the door for numerous questions about trisections which arise from analogous questions about Heegaard splittings. Trisections give a new structure to smooth 4–manifolds which offer new insight into the elusive nature of dimension 4.

Relative trisections were first introduced by Gay and Kirby in [3] for compact manifolds with connected boundary. This work extends trisections to 4–manifolds with $m > 1$ boundary components, completing the theory to all smooth, compact, connected, oriented 4–manifolds. Several results of Gay and Kirby immediately extend to this more general setting, such as Lemma 2 which tells that a relative trisection of X induces an open book decomposition on *every* boundary component of ∂X.

The gluing theorem in Section 3.2 reveals the compatibility between the structures of closed and relative trisections. The idea behind gluing trisections is very simple. Suppose two 4–manifolds have non-empty, diffeomorphic boundaries. One would hope that the closed 4–manifold $X = W \underset{\partial}{\cup} Z$ inherits some structure from W and Z. Theorem 6 tells us that trisections are respected under such an operation, so long as the induced open book decompositions on ∂Z and ∂W are respected by this gluing. Moreover, we can generalize this to gluing manifolds along proper subsets of their boundaries to induce a *relative trisection* on the resulting manifold with boundary.

Central to the theory of trisections is the notion of stabilization. Much like that of Heegaard splittings, a stabilization of a closed trisection allows us to alter the trisection data of a fixed smooth 4–manifold. The closed stabilization defined in [3] increases the number of $S^1 \times B^3$ summands by one, but increases the genus of the trisection surface by 3. This stabilization can be thought of as taking the connected sum of $(X, \mathcal{T})$ with the standard genus 3 trisection of S^4. This stabilization can be done to a relative trisection, but it does not alter the boundary data. In Section 3.3, we introduce a *relative stabilization* which stabilizes both the trisection and the induced open book on ∂X. This allows us to strengthen the uniqueness statement for relative trisections in [3].

Chapter 2

Preliminaries

2.1 Notation and Conventions

We will always assume our manifolds are smooth, compact, connected, and oriented unless stated otherwise. We use the following notation throughout:

- $F_{g,b}$ genus g surface with b boundary components.

- F_g, Σ_g closed genus g surface.

2.2 Heegaard Splittings: 3–dimensional Inspiration

Definition 1. A *genus g Heegaard splitting* of a 3–manifold M is a decomposition $M = H_1 \cup H_2$ into two diffeomorphic 3–dimensional handlebodies $H_1 \cong H_2 \cong \natural^g S^1 \times D^2$, which intersect along their common boundary $\partial H_1 \cong \partial H_2 \cong H_1 \cap H_2 \cong \Sigma_g$.

It is not difficult to prove that every 3–manifold admits a Heegaard splitting. Let $f : M \to \mathbb{R}$ be a self indexing Morse function, i.e., the image of an index i critical point is i. Then M decomposes as $f^{-1}([0, 3/2]) \cup f^{-1}([3/2, 3])$ with Heegaard surface $f^{-1}(3/2) \cong \Sigma_g$, where g is the number of index 1 critical points of f.

A crucial component to trisections and Heegaard splittings is the notion of a stabilization, whereby we obtain a new decomposition from an old one in the most trivial way. Given a genus g Heegaard splitting $M = H_1 \cup H_2$ with $H_1 \cap H_2 = \Sigma_g$, let $\alpha \subset H_2$ be a boundary parallel, properly embedded arc and let $N \subset H_2$ be a neighborhood of α. We obtain the genus $g+1$ Heegaard splitting $M = H_1' \cup H_2'$ where $H_1' = \overline{H_1 \cup N}$, $H_2' = H_2 \setminus N$, and $H_1' \cap H_2'$ is a genus $g+1$ surface. This process, shown in Figure 2.2, is called *stabilization* and (Σ', H_1', H_2') is called the *stabilization* of (Σ, H_1, H_2).

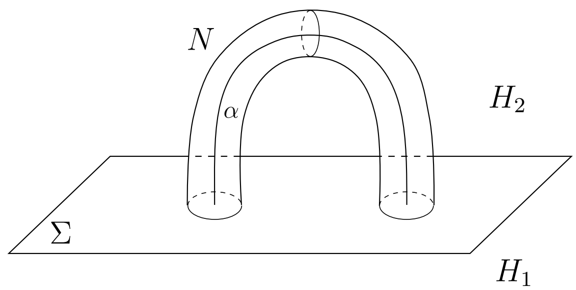

Figure 2.1: Stabilizing a Heegaard Splitting

The following theorem is the first of several results in Heegaard splittings for which there are trisection analogs.

Theorem 1. [Reidemeister-Singer [9, 10]] Any two Heegaard splittings of the same manifold become isotopic after sufficiently many stabilizations.

Example 1 (Heegaard Splitting of $S^1 \times S^2$)**.** Figure 1 shows two different Heegaard splittings of $S^1 \times S^2$, where the outer sphere is identified with the inner sphere. On the left is a genus 1 splitting with Heegaard surface Σ given by $S^1 \times \gamma$, where $\gamma \subset S^2$ is a great circle. In the figure Σ appears as an annulus.

On the right is a stabilization of the previous Heegaard splitting, which has increased the genus by 2.

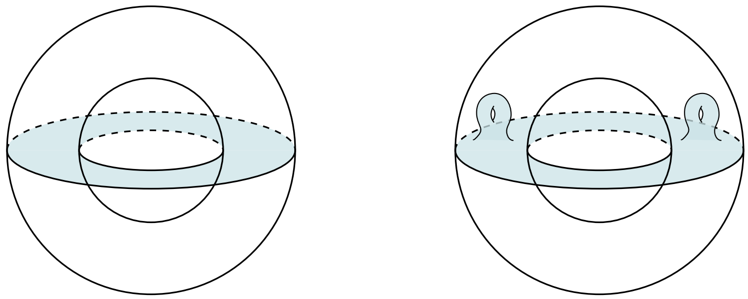

Figure 2.2: Two Heegaard Splittings of $S^1 \times S^2$

2.3 Open Book Decompositions of 3-manifolds

Open book decompositions have significantly contributed to our knowledge of 3–manifolds. Most notably, is perhaps the Giroux correspondence, which states the following: There is a one-to-one correspondence between isotopy classes of oriented contact structures on M^3 and open book decompositions of M up to positive Hopf stabilization [4]. John Etnyre gives a beautiful treatment of open book decompositions and contact structures in [2]. We include only the basic definitions and results, which is all that is necessary for our purposes.

Definition 2. An *open book decomposition* of a connected 3–manifold M is a pair (B, π) such that B is a link in M called the *binding* and $\pi : M \setminus B \to S^1$ is a fibration such that the closure of the fibers $\overline{\pi^{-1}(t)} = \Sigma_t$, called *pages*, are genus g surfaces with $\partial \Sigma_t = B$ for every t.

When M is disconnected, an open book decomposition of M is an open book decomposition of each component, and thus has disconnected pages $P = \bigsqcup_{i=1}^{m} P_i$, where P_i is a genus p_i surface with b_i boundary components. We denote $p = \sum_{i=1}^{m} p_i$ and $b = \sum_{i=1}^{m} b_i$

Theorem 2 (Alexander [1])**.** Every closed, oriented 3–manifold has an open book decomposition.

Definition 3. An *abstract open book* is a pair (Σ, ϕ), where Σ is a surface with boundary and $\phi \in \mathit{Diff}_+(\Sigma, \partial\Sigma)$, the orientation preserving diffeomorphisms of Σ which fix $\partial\Sigma$ pointwise.

Given an abstract open book, we can construct a closed 3–manifold M_ϕ as follows. First construct the mapping torus $\Sigma_\phi = \Sigma \times I/((x,0) = (\phi(x),1))$. For each boundary component of Σ, there is a torus boundary component of Σ_ϕ. Each of these torus boundaries can be filled with a copy of $S^1 \times D^2$ to obtain a closed 3–manifold. By requiring $\partial\Sigma \times \{t\} \subset \Sigma_\phi$ to be attached along $S^1 \times \{t\} \subset S^1 \times \partial D^2 \subset S^1 \times D^2$, we have constructed a unique closed 3–manifold

$$M_\phi = \Sigma_\phi \cup \left(\bigsqcup_{|\partial\Sigma|} S^1 \times D^2 \right),$$

with an open book decomposition (B_ϕ, π_ϕ), where $B_\phi \cong \partial\Sigma$ and pages $P_\phi \cong \Sigma$.

The following lemma gives standard facts about open books and allows us to move freely between abstract and non-abstract open books.

Lemma 1. Every open book (B, π) of M corresponds to an abstract open (P, ϕ) such that $M_\phi \cong M$. Additionally, M_ϕ and (B_ϕ, π_ϕ) determined up to diffeomorphism by (P, ϕ).

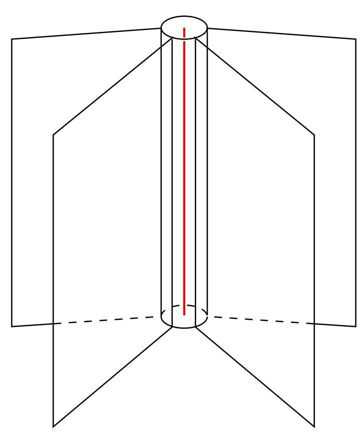

Figure 2.3: The Trivial Open Book Decomposition of S^3

Example 2. We can define a fibration $f : S^3 \setminus N \to S^1$, where N is a neighborhood of the binding $B = z - \text{axis} \cup \{\infty\}$, such that the fibers are $f^{-1}(\theta) \cong D^2$. This is known as the trivial open book decomposition of S^3 and is depicted in Figure 2.3 with the binding drawn in red.

Example 3. Figure 3 depicts an open book decomposition of $S^1 \times S^2$, where we should be thinking of the outer sphere being identified with the inner sphere. The page is an annulus and binding B is a two component unlink shown as the red lines $\{x_N\} \times S^1$ and $\{x_S\} \times S^1$, where $x_N, x_S \in S^2$ are the north and south poles respectively.

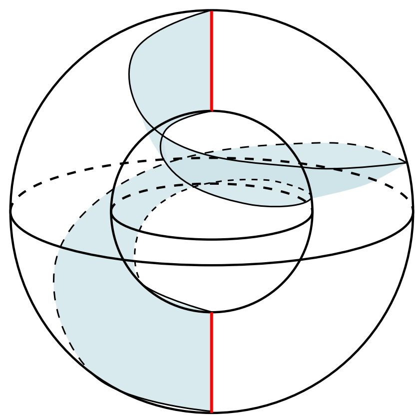

Figure 2.4: Example of an Open Book Decomposition of $S^1 \times S^2$

There is a notion of stabilizing an open book decomposition which is of paramount importance to open books, as well as trisections. Given an abstract open book, (P, ϕ), choose a properly embedded arc $\alpha \subset P$. Attach a 2–dimensional 1–handle to $\partial\alpha \times I \subset \partial P$, giving a new surface P'. The co-core of the 1–handle together with α comprise a simple closed curve $\gamma \subset P'$, which we require to have page framing -1. Define the new abstract open book $(P', \tau_\gamma \circ \phi)$. This process is called a *Hopf stabilization* of (P, ϕ). It is a standard result that $M_\phi \cong M_{\tau_\gamma \circ \phi}$. The page P' can also be viewed as the result of plumbing a Hopf band of onto P. The top of Figure 2.4 depicts how a Hopf stabilization changes the pages of an open book.

We have the following uniqueness theorem for open book decompositions.

Theorem 3 (Giroux [5]). Every open book decomposition of a rational homology sphere can be made isotopic after some number of positive Hopf stabilizations.

2.4 Lefschetz Fibrations

The following section addresses the topology of Lefschetz fibrations and follows the expositions in Section 8.2 of [6] and Section 10.1 of [8].

Definition 4. Let X be a compact, oriented 4–manifold. A *Lefschetz fibration* on X is a map $f : X \to S$, where S is a compact , connected, oriented surface, such that

i) f has finitely many critical points $\Gamma = \{p_1, \ldots, p_n\} \subset int(X)$ such that $f(p_i) \neq f(p_j)$ for $i \neq j$

ii) around each critical point f can be locally modeled by an orientation preserving chart as $f(u, v) = u_1^2 + v_2^2$.

iii) in the complement of the singular fibers, $f^{-1}(f(\Gamma))$, f is a smooth fibration

The fibers of critical values are said to be *singular* and all other fibers are *regular*. Removing the condition that charts preserve orientation results in what is known as an *achiral Lefschetz fibration.*

Note that if regular fibers are closed surfaces, then $\partial X = f^{-1}(\partial S)$. However, in the case of bounded fibers we have that ∂X consists of $f^{-1}(\partial S)$ together with a neighborhood of the boundary of a regular fiber. For our purposes, we will only consider Lefschetz fibrations over D^2 with bounded fibers.

Since singular fibers are defined locally by $f(u, v) = u^2+v^2$, a regular fiber in a sufficiently small neighborhood of a singular point p_i is given by $u^2 + v^2 = t$, for some $t > 0$ (multiply by a complex number to obtain $t \in \mathbb{R}$.) If we consider the intersection of the fiber F_t with $\mathbb{R}^2$, then we obtain the equation $x_1^2 + x_2^2 = t$ (where $u = x_1 + iy_1$ and $v = x_2 + iy_2$). This equation defines a circle $\gamma_i \subset F_t$ which bounds a disk $D_t \in \mathbb{R}^2$. γ_i is called the *vanishing cycle* of the critical point p_i. As t approaches 0, $\gamma_i = \partial D_t$ contracts to a point and creates our singular fiber. Thus, a neighborhood of a singular fiber is obtained from attaching a neighborhood of a disk D_t to a neighborhood of a regular fiber νF_t. This neighborhood is in fact a 4–dimensional 2–handle attached along the vanishing cycle γ_i. It can be shown that γ_i has framing -1 relative to F_t (cf. [6] [8]). In the case of an achiral Lefschetz fibration, γ_i will have a relative page framing of ± 1, depending on whether the chart reverses or preserves orientation.

Lefschetz fibrations allow us to recover the topology of X from its vanishing cycles. It is well known that the monodromy of f in a neighborhood of a critical value q_i is given by a right-handed Dehn twist along the vanishing cycle γ_i. We can use this fact to understand the global monodromy of f.

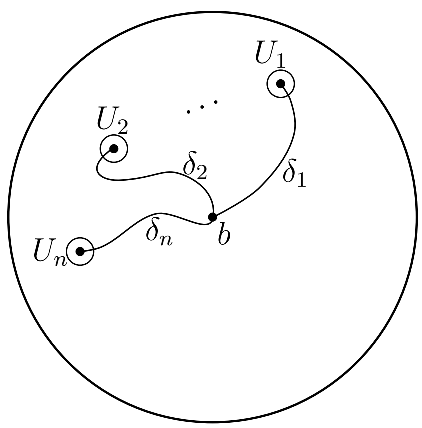

Figure 2.5: Determining the monodromy from vanishing cycles

Fix a regular value q of $f : X \to D^2$ and an identification $f^{-1}(q) \cong F_{g,b}$. Let U be an open neighborhood of b which does not contain any critical values, and let U_i be open neighborhoods of q_i such that $U_i \cap C = \{q_i\}$. Let δ_i be a smooth path from q to q_i missing all other critical values, and let us further assume that they are enumerated so that they appear in increasing order when traveling counter clockwise around q as in Figure 2.4. First, consider

$$X_1 = f^{-1}\left(U_0 \cup \nu\delta_1 \cup U_1\right).$$

As discussed above, $X_1 \cong F_{g,b} \times D^2 \cup H_1$, where H_1 is a 2–handle attached along γ_1 with framing -1 relative to $F_{g,b}$. Moreover, f induces an open book decomposition of ∂X_1 with pages $F_{g,b}$ and monodromy τ_{γ_1}, a Dehn twist along γ_1. We can now extend our set to X_2 by including $\nu\delta_2$ and U_2. Thus, X_2 is diffeomorphic to $\left(F_{g,b} \times D^2\right) \cup H_1 \cup H_2$, where H_2 is a 2–handle attached along γ_2 with the usual framing. The monodromy of the induced open book on ∂X_2 is $\tau_{\gamma_2} \circ \tau_{\gamma_1}$. We can continue this process of including neighborhoods of critical points, each time adding a 2–handle along the associated vanishing cycle and modifying the previous monodromy by post-composing with the appropriate Dehn twist. The final step in this process gives

$$X_n := f^{-1}\left(U \cup \left(\bigcup_{i=1}^{n} \nu\delta_1\right) \cup \left(\bigcup_{i=1}^{n} U_i\right)\right).$$

Note that since $X \setminus X_n$ contains no critical values, we have that $X \cong X_n$. Therefore, we have that $X \cong \left(F_{g,b} \times D^2\right) \cup \left(\bigcup_{i=1}^{n} H_i\right)$, where each H_i is attached along γ_i, giving us $\chi(X) = \chi(F_{g,b}) + n$. Moreover, part of ∂X_n, and thus part of ∂X, is a fibration over over S^1 with monodromy $\phi = D_{\gamma_n} \cdots D_{\gamma_1}$, called the *global monodromy* of f. This fibration is the induced open book decomposition on ∂X.

As we will see in Section 3.3, a relative trisection of X also induces an open book decomposition on ∂X. The following modification of a Lefschetz fibration will allow us to define a stabilization of relative trisections.

Let $f : X \to D^2$ be a Lefschetz fibration of X^4 with bounded fibers and critical set Γ. We can obtain a new Lefschetz fibration $\tilde{f} : X \to D^2$ by adding a 4–dimensional canceling $1-2$ pair as follows: We attach the 1–handle H^1 so that the attaching sphere (S^0) lies in the

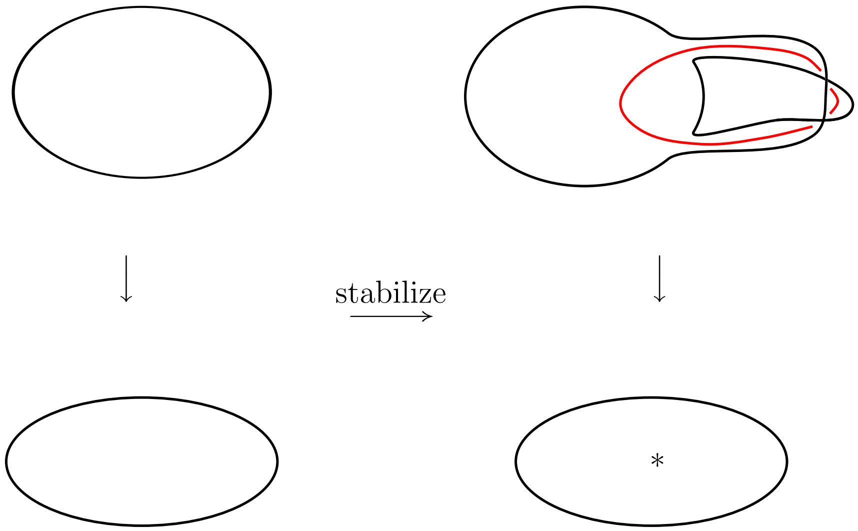

Figure 2.6: Stabilizing a Lefschetz fibration inducing a stabilization of the open book

binding of the open book decomposition of ∂X induced by f. This allows us to extend f to a Lefschetz fibration $\overline{f} : X \cup H^1 \to D^2$ with regular fibers $\overline{f}^{-1}(y) = f^{-1}(y) \cup 1$–handle. We now attach the canceling 2-handle H^2 along a simple closed curve γ which lies in a single regular fiber with framing ± 1 relative to the fiber. After smoothing corners, we have a Lefschetz fibration $\tilde{f} : X \to D^2$ whose regular fibers are equal to that of $\overline{f}$ and whose critical set $\widetilde{\Gamma} = \Gamma \cup \{y\}$. Note that this may give an achiral Lefschetz fibration. By following the above process of obtaining a handle decomposition and global monodromy of X, one finds that the induced open book decomposition $\tilde{f}|_{\partial X}$ is a Hopf stabilization of $f|_{\partial X}$.

2.5 Trisections of Closed 4–manifolds

Definition 5. [3] A (g, k)*–trisection* of a closed 4–manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

i) for each i, X_i is diffeomorphic to $\natural^k S^1 \times B^3$,

ii) for each i, $(X_i \cap X_{i+1}) \cup (X_i \cap X_{i-1})$ is a genus g Heegaard splitting of ∂X_i,

where indices are taken mod 3.

As a consequence, the triple intersection $X_1 \cap X_2 \cap X_3 = F_g$ is a genus g surface, called the *trisection surface.* Additionally, a handle decomposition of X tells us that $\chi(X) = 2+g-3k$. This tells us two things. The first is that for any given manifold X, k is determined by g which allows us to refer to a (g, k)-trisection as a *genus g trisection.* The second fact is that the genera of any two trisections of a fixed X must be equivalent mod 3. We will occasionally denote a trisection of X by $\mathcal{T}_X$, or $\mathcal{T}$.

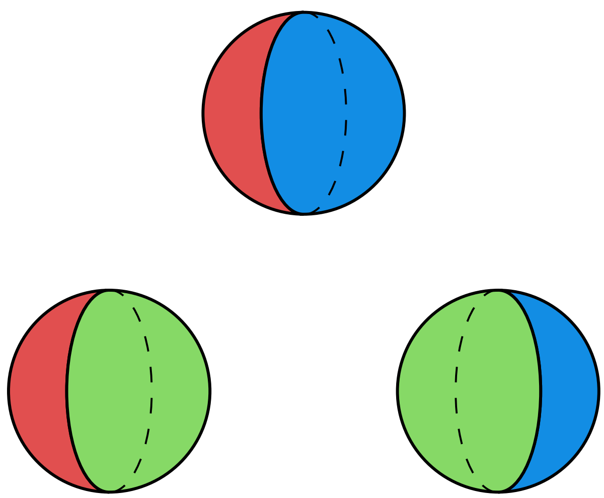

Figure 2.7: Schematic for $\mathcal{S}_0$, the Trivial Trisection of S^4

Example 4 (Trivial Trisection of S^4)**.** The genus 0 trisection of S^4 is found by viewing $S^4 \subset \mathbb{C} \times \mathbb{R}^3$ and explicitly dividing it up into three pieces: $X_j = \{(re^{i\theta}, x_2, x_3, x_4) | 2\pi j/3 \leq \theta \leq 2\pi(j+1)/3\}$. This gives us $g = 0$, and hence $k = 0$. Thus, for each i, $X_i \cong \natural^0 S^1 \times B^3 = B^4$ and $\partial X_i \cong S^3$. Since $X_i \cap X_{i-1}$ and $X_i \cap X_{i+1}$ must be handlebodies for a genus 0 Heegaard splitting of X_i, we see that $X_i \cap X_j \cong B^3$ and $X_1 \cap X_2 \cap X_3 \cong S^2$.This gives the following:

- $X_i \cong B^4$
- $X_i \cap X_j \cong B^3$
- $X_1 \cap X_2 \cap X_3 \cong S^2$

Figure 4 gives us a visualization for this trisection in three dimensions. The colored hemispheres tell us how to glue each of the X_i's to one another. Under these identification, the result is S^4. (In this toy picture the result would be S^3.) Note that the triple intersection of the X_i's can be seen on each ball as the great circle separating the colors. Additionally, notice that the triple intersection, S^2 (depicted in Figure 4 as S^1) is closed.

The stabilization move in the trisection setting is a bit more complex than in the Heegaard splitting setting. We still wish to obtain a new trisection $\mathcal{T}'$ by modifying $\mathcal{T}$ in the most trivial way possible. Choose a boundary parallel, properly embedded arc $\alpha \subset X_2 \cap X_3$ and a regular neighborhood $N_1 \subset X_2 \cap X_3$ of α. Choose arcs β, γ and their neighborhoods $N_2 \subset X_1 \cap X_3$ and $N_3 \subset X_1 \cap X_2$ similarly. We define the pieces of our new trisection to be

$$X_1' := \overline{X_1 \cup N_1} \setminus (N_2 \cup N_3)$$
$$X_2' := \overline{X_2 \cup N_2} \setminus (N_1 \cup N_3)$$
$$X_3' := \overline{X_3 \cup N_3} \setminus (N_1 \cup N_2)$$

Attaching the 1–handles N_i to X_i results in the boundary connected sum with $S^1 \times B^3$. However, removing the other two neighborhoods from X_i do not change its topology. This is due to the fact that each curve lies in the intersection of two pieces of our trisection. This has the effect of "digging a trench" out of X_i. On the other hand, each one of these neighborhoods are attached to the trisection surface which increases the genus of the trisection by three. This should be expected from the equation $\chi(X) = 2 + g - 3k$.

Definition 6. The above process is called the *closed or interior stabilization* of the trisection $\mathcal{T}$.

The resulting stabilized trisection is shown in Figure 2.5. Note that an interior stabilization of a (g, k)–trisection is a $(g + 3, k + 1)$–trisection.

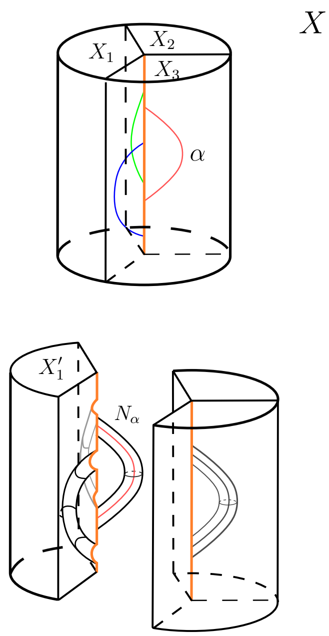

Figure 2.8: Stabilizing a Trisection

The following theorem is the trisection analog of Theorem 1.

Theorem 4 (Gay-Kirby, 2012 [3])**.** Every smooth, closed, connected, oriented 4–manifold admits a trisection. Moreover, any two trisection of the same 4–manifold become isotopic after a finite number of stabilizations.

Chapter 3

Relative Trisections

3.1 Basic Definitions and Theorems

Just as in the closed case, a trisection of a 4–manifold X with non-empty boundary is a decomposition $X = X_1 \cup X_2 \cup X_3$ where $X_i \cong \natural^k S^1 \times B^3$, for some k, such that the X_i's have "nice" intersections. Before the proper definition can be stated, we will discuss the model piece to which each X_i, and their intersections, will be diffeomorphic.

We begin with $F_{g,b}$, a connected genus g surface with b boundary components, and attach n 3–dimensional 2–handles to $F_{g,b} \times \{1\} \subset F_{g,b} \times [0,1]$ along a collection of n essential, disjoint, simple, closed curves. If ∂X has m connected components, then we require the curves to separate $F_{g,b}$ into m components, none of which are closed. Further, the curves cannot be such that surgering along them results in closed components. Such a 3–manifold C is a *compression body*. We define our model pieces $Z = C \times I \cong \natural^k S^1 \times B^3$, where $k = 2g + b - 1 - n + m - 1$.

Remark 1. In general, a *compression body* is a 3–manifold which is the result of attaching 0 and 1 handles to $\Sigma \times I$, where Σ is a compact surface, with or without boundary. In what follows we will only be dealing with compression bodies such as C.

It is sometimes convenient to consider a Morse function $f : C \to [0,1]$ with $f^{-1}(0) = F_{g,b}$ and $f^{-1}(1) = P$, the "other end" of our compression body. We will use the notation in Section 2.3, $P = \bigsqcup_{i=1}^{m} P_i$, where $P_i \cong F_{p_i,b_i}$, $\sum_i p_i = g - n + m - 1$ and $\sum_i b_i = b$. The function f will only have n index–2 critical points. Let us arrange for $m-1$ separating handles to have the same critical value λ, and the remaining critical values to be distinct and strictly less than λ. A schematic for this construction is given in Figure ; the red lines represent the critical levels of a Morse function)

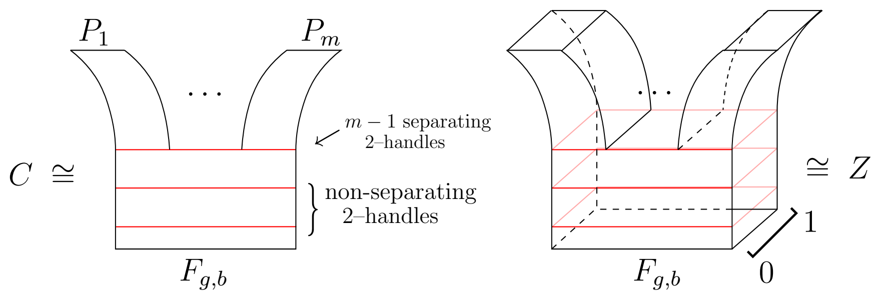

Figure 3.1: Constructing the Model Pieces

Note that by constructing C upside down, it becomes immediately clear that C is a 3–dimensional handlebody: We attach n 3–dimensional 1–handles to $P \times I$, ensuring to connect every component. Since $P \times I$ is a neighborhood of a punctured surface, we have that $P_i \times I \cong \natural^{l_i} S^1 \times D^2$, where $l_i = 2p_i + (b-1)$. Thus, attaching 1–handles in the prescribed manner gives us

$$C \cong \natural^k S^1 \times D^2,$$

where $k = n + (m-1) + \sum_i l_i$. (We will regularly make use of the fact that our compression bodies are 3–dimensional handlebodies. This is due to the fact that $F_{g,b}$ has non-empty boundary.) Thus,

$$Z \cong \natural^k S^1 \times B^3.$$

Consider ∂Z, which we will decompose into two pieces,

$$\begin{aligned} In(\partial Z) &:= \quad (C \times \{0\}) \cup (F_{g,b} \times I) \cup (C \times \{1\}) \\ Out(\partial Z) &:= \quad (\partial F_{g,b} \times I \times I) \cup (P \times I) \end{aligned}$$

called the *inner and outer boundaries of* Z as in Figure . $In(\partial Z)$ is the portion of ∂Z which gets glued to the other pieces in the trisection, whereas $Out(\partial Z)$ contributes to ∂X.

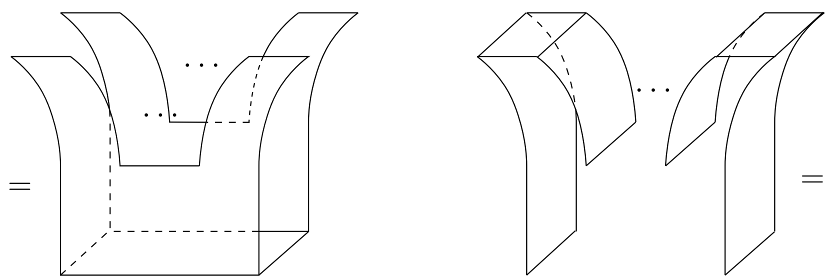

Figure 3.2: Decomposing ∂Z

There is a standard *generalized Heegaard splitting* of $In(\partial Z)$, i.e., a decomposition of a 3–manifold with boundary $M = C_1 \cup C_2$, where $C_1 \cong C_2$ are compression bodies which intersect in surface with boundary. We decompose $In(\partial Z)$ as

$$In(\partial Z) = (C \times \{0\} \cup F_{g,b} \times [0, 1/2]) \cup (F_{g,b} \times [1/2, 1] \cup C \times \{1\}) .$$

which we will denote as $In(\partial X_1) = Y_0^+ \cup Y_0^-$, where $Y_0^+ \cap Y_0^- \cong F_{g_0,b}$. We further stabilize this splitting (on the interior of the surface) some number of times (possibly zero) which increases the genus of the splitting. We will denote this stabilized, standard splitting as $In(\partial Z) = Y^+ \cup Y^-$, where $Y^+ \cap Y^- = F_{g,b}$. It should be noted that the stabilizations involved do not alter the 4–manifold Z_i in any way; only the decomposition of the 3–manifold $In(\partial Z)$.

The "nice intersections" mentioned earlier can now be defined: $X_i \cap X_{i+1} \cong Y^+$ and $X_i \cap X_{i-1} \cong Y^-$. Alternately phrased, $(X_i \cap X_{i+1}) \cup (X_i \cap X_{i-1})$ is this particular generalized

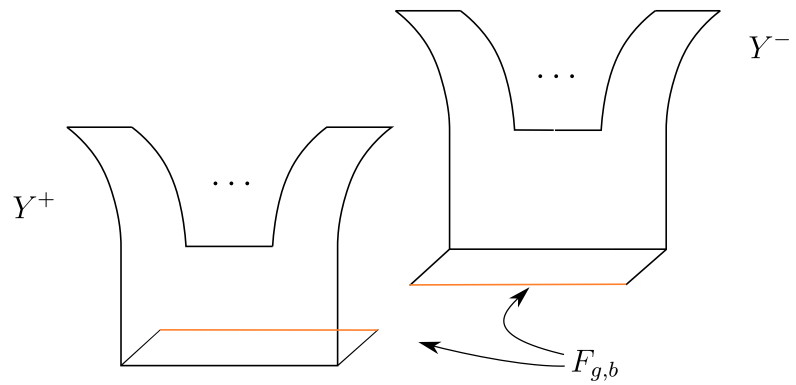

Figure 3.3: Decomposing $In(\partial Z)$ as $Y^+ \cup Y^-$

Heegaard splitting of $In(\partial X_i) \cong In(\partial Z_i)$. We now give the proper definition using the above notation

Definition 7. A *relative trisection* of a smooth 4–manifold with boundary is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that, for some Z with splitting $In(\partial Z) = Y^+ \cup Y^-$ constructed as above

i) for each i there exists a diffeomorphism $\varphi_i : X_i \to Z$,

ii) for each i, we have $\varphi_i(X_i \cap X_{i+1}) = Y^+$ and $\varphi_i(X_i \cap X_{i-1}) = Y^-$

where indices are taken mod 3. We will sometimes denote a trisection of X as $\mathcal{T}_X$, or $\mathcal{T}$.

As a consequence, the triple intersection $X_1 \cap X_2 \cap X_3 = F_{g,b}$ is a surface with boundary called the *trisection surface*, and the outer boundaries comprise ∂X. Let us denote $Out(\partial X_i) = \varphi_i^{-1}(Out(\partial Z))$.Note $Out(\partial X_i) = X_i \cap \partial X$. The connected components of $Out(\partial X_i)$ are given by $P_i \times I$ together with $\nu\partial P_i$, a 3–dimensional neighborhood of ∂P_i. Thus, gluing the X_i's to one another induces a fibration $\partial X \setminus \nu\partial P \to S^1$ with fiber P. In other words, (P, ϕ) is an abstract open book corresponding to ∂X, where ϕ is determined by the attaching maps $\{\varphi_i\}$.

We have thus proved the following lemma, which generalizes Gay and Kirby's [3] result to smooth, compact 4–manifolds with an arbitrary number of boundary components.

Lemma 2. A relative trisection of X induces an open book decomposition of ∂X.

Example 5. The simplest relative trisection is the trivial trisection of B^4, where

- $X_i \cong B^4$

- $X_i \cap X_{i+1} \cong B^3$

- $X_1 \cap X_2 \cap X_3 \cong D^2$

We can more easily visualize this in three dimensions. Figure 5 represents the pieces of our trisection in dimension three. As stated above, $X_i \cong B^4$. The colored regions on a

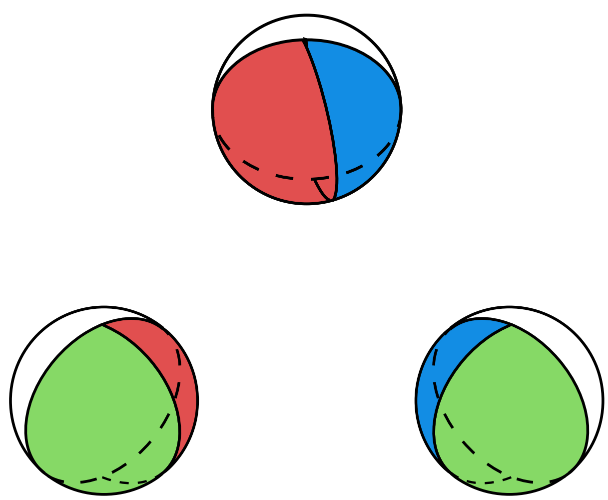

Figure 3.4: Trivial Trisection $\mathcal{B}$ of B^4

given X_i comprise $In(\partial X_i)$ (which are modeled by D^2). We then take a genus–0 generalized Heegaard splitting of $In(\partial X_i) \cong B_i^{3+} \cup_{D^2} B_i^{3+}$. Each $B_i^{3\pm}$ is colored so as to indicate where

X_i will glue to X_j. Taking indices mod 3, we trivially glue B_i^{3+} to B_{i+1}^{3-}. Doing so yields $B^4 \cong X_1 \cup X_2 \cup X_3$. Moreover, we see that $Out(\partial X_i) \cong B^3$ and our gluing gives us

$$\partial X = \bigcup_i Out(\partial X_i) \cong S^3.$$

Notice that the triple intersection has boundary. In Figure , it is represented by the arc (B^1) which separates each color on the inner boundaries. As one might expect, the induced open book $\mathcal{B}|_{\partial B^4}$ is the trivial open book on S^3.

Example 6 (Relative Trisection of $S^3 \times I$)**.** Consider the compression body C obtained by attaching a 3–dimensional 2–handle to $(S^1 \times I) \times I$ along $(S^1 \times \{1/2\}) \times \{1\}$. Notice that $C \cong D^2 \times I$, and thus $Z = C \times I \cong B^4$. Consider the standard unstabilized splitting $In(\partial Z) = Y^+ \cup Y^-$, where $Y^+ \cong Y^- \cong C$. We then attach the pieces to one another by mapping a Y^+ of one piece to the Y^- of the other via the identity map. To see that this defines a trisection, we note that the 4–dimensional 1–handles (in X_i) glue together to give $D^1 \times S^3$. That is, each X_i is diffeomorphic to $D^1 \times B^3$ whose attaching maps identify points in the B^3 components of these handles. This gluing is shown explicitly in Figure 4, where we no longer view this as a schematic. This trisection can be viewed as a construction, or as the (interior) connected sum of two copies of $(B^4, \mathcal{B})$.

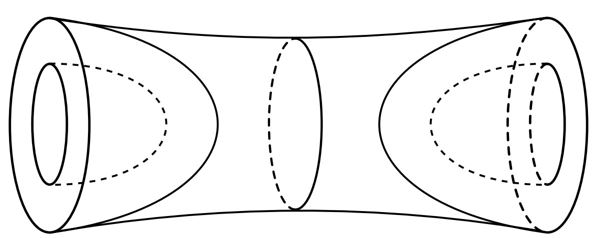

Figure 3.5: Compression body $C \cong X_i \cap X_j$ for $S^3 \times I$

The previous example shows that $S^3 \times I$ can be trisected into three 4–dimensional 1–handles. In the same spirit, we can take the connected sum of two trisected manifolds $(X, \mathcal{T})$

and $(X', \mathcal{T}')$ to obtain $(X\#X', \mathcal{T}\#\mathcal{T}')$, where the pieces of the new trisection are X_i and X_i' connected via 1–handle.

Theorem 5. Let X be a smooth, compact, connected 4–manifold with boundary such that each connected component of ∂X is equipped with a fixed open book decomposition. There exists a trisection of X which restricts to ∂X as the given open books.

Proof. Let (B, ϕ) be an open book decomposition of ∂X with page P. If ∂X has m connected components, then so does P. We will use the given boundary data to construct a Morse function $f : X \to I$.

Extend $\phi : \partial X \setminus \nu B \to S^1$ to the whole of ∂X, $\phi : \partial X \to D^2$ by $(x, z) \mapsto z$ for every $(x, z) \in B \times D^2$. Then fix an identification of D^2 with $I \times I$ and compose ϕ with projection onto the first factor, giving us a smooth map $f : \partial X \to I$ such that

i) $f^{-1}(0) \cong \bigsqcup_{i=1}^{m} (P_i \times I) \cong f^{-1}(1)$

ii) $f^{-1}(t) \cong \bigsqcup_{i=1}^{m} (P_i \times \{0\} \cup (\partial P_i \times I) \cup P_i \times \{1\}) \;\; 0 < t < 1.$

Extend f to a Morse function on all of X and consider the handle decomposition given by f. Notice that since X is connected, such a function necessarily admits 1–handles. Let h_i denote the number of i–handles. Without loss of generality, we can assume that the handles are ordered by index. Moreover, by adding canceling pairs we can arrange for $h_1 = h_3$.

Let $\varepsilon, a \in (0, 1)$ be such that $f^{-1}([0, \varepsilon]) = f^{-1}(0) \times I$ and $[\varepsilon, a]$ contains all of the index 1 critical values, but no others. Define $X_1 = f^{-1}([0, a])$. We then have

$$\begin{aligned} f^{-1}([0,\varepsilon]) &\cong f^{-1}(0) \times I \\ &\cong (P \times I) \times I \\ &\cong \bigsqcup_{i=1}^{m} \left(\natural^{l_i} S^1 \times B^3\right), \end{aligned}$$

where $l_i = 2p_i + (b_i - 1)$. Connectedness gives us

$$\begin{aligned} X_1 &\cong f^{-1}([0,\varepsilon]) \cup 1\text{-handles} \\ &\cong \natural_{i=1}^{m} (\natural^{l_i} S^1 \times B^3) \natural (\natural^{h_1-m+1} S^1 \times B^3) \\ &\cong \natural^{l+k_1-m+1} S^1 \times B^3, \end{aligned}$$

where $l = \sum_i l_i$.

We will now give $f^{-1}(a)$ a Heegaard splitting: Define $N = f^{-1}(\varepsilon)$ and $M = \overline{\partial(f^{-1}([0,\varepsilon])) \setminus N}$. It is not hard to see that $M \cong N \cong P \times I$, and thus there is a natural generalized Heegaard splitting of $N \cong (P \times [0, 1/2]) \cup (P \times [1/2, 1])$. Thus when we attach the 1–handles, some of which connect components to each other, we have a sort of "unbalanced" decomposition $\partial X = M \cup N'$, where N' is diffeomorphic to

$$(F_{p,b} \times [0, 1/2]) \natural H_1 \bigcup (F_{p,b} \times [1/2, 1]) \natural H_2$$

and $\#^{k_1-m} S^1 \times S^2 = H_1 \cup H_2$ is the standard genus $h_1 - m$ Heegaard splitting. Note that $N' \cong f^{-1}(a)$. Let us denote this generalized Heegaard splitting $N' = Y^+ \cup Y^-$, where Y^+ and Y^- are compression bodies from $F_{p+h_1-m+1,b}$ to P which intersect along the surface of greater genus.

Let $L \subset N'$ be the framed link which corresponds to the attaching spheres of the 2–handles given by f. Project L onto the splitting surface $F_{g,b} = Y^+ \cap Y^-$ in such a way that each component of L non-trivially contributes to the total number of self intersections, or crossings, denoted by c. This can be done by Reidermeister 1 moves if necessary.

We first consider the special case where $c = h_2$. Stabilizing the generalized Heegaard splitting $Y^+ \cap Y^-$ at every crossing resolves the double points by providing 1–handles whose co-cores intersect a unique link component exactly once. Additionally, every link component has such an intersection by construction. We also wish for the framings of the now embedded link L to be consistent with the page framings. This is achieved by adding a kink via a Reidermeister 1 move and stabilizing at the new crossing. This changes the page framing by ± 1, which will allow us to achieve any framing through this process. Although we may have stabilized several times, we still denote $N' = Y^+ \cup Y^-$ with $Y^+ \cap Y^- = F_{g,b}$.

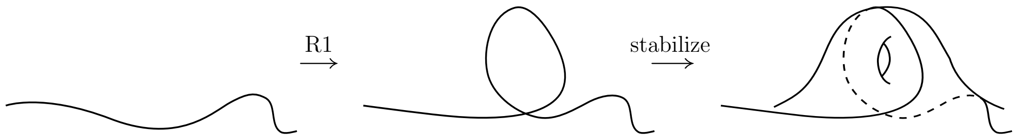

Figure 3.6: Changing the Page Framing by ± 1

Let us now define X_2 to be a collar neighborhood of Y^+ in the complement of X_1 together with the 2–handles of X. It is important that in X_2, the 3–dimensional 1–handles of Y^+ give rise to 4–dimensional 1–handles of X_2. Since we have just arranged for the attaching sphere of each 2–handle to intersect the co-cores of the 1–handles, there are $c = h_2$ canceling $1 - 2$ pairs in X_2. (We can slide 1–handles over one another to obtain a one-to-one correspondence between 2–handles and co-cores of 1–handles).

We can now verify that X_2 is a handlebody of the appropriate genus. First, we have that

$$\begin{aligned} Y^+ &\cong (F_{p,b} \times I) \natural H_1 \\ &\cong \left(\natural^l S^1 \times D^2\right) \natural \left(\natural^{h_1 - m + 1 + c} S^1 \times D^2\right) \\ &\cong \natural^{l + h_1 - m + 1 + c} S^1 \times D^2, \end{aligned}$$

where we have taken the view that the c stabilizations occur in $H_1 \cup H_2$. Finally, since we have arranged for the 2-handles to cancel 1–handles, we obtain the desired result:

$$\begin{aligned} X_2 &\cong Y^+ \times [0, \varepsilon] \bigcup_L 2 - \text{handles} \\ &\cong \left(\natural^{l + h_1 - m + 1 + c} S^1 \times B^3\right) \bigcup_L 2 - \text{handles} \\ &\cong \natural^{l + h_1 - m + 1} S^1 \times B^3, \end{aligned}$$

Finally, define $X_3 := \overline{X \setminus (X_1 \cup X_2)}$. Since $h_1 = h_3$, "standing on your head" gives us a$X_1 \cong X_3$.

As for the intersections, $X_1 \cap X_2 \cong Y^+$ and $X_1 \cap X_3 \cong Y^-$ by definition. To see $X_2 \cap X_3 \cong Y^+$ we exploit the one-to-one correspondence between link components and a subset of the co-cores of 1–handles of Y^+. Each surgery on $Y^+ \times \{1\}$ defined by a link component of $L \subset int(Y^+ \times \{1\})$ can be done in a unique $S^1 \times D^2$ summand of $Y^+ \times \{1\} \cong \natural S^1 \times D^2$. Such a surgery on $S^1 \times D^2$ results in $S^1 \times D^2$ and simply changes which curve bounds a disk. Thus, the surgery 3–manifold $(Y^+ \times \{1\})_L$ is diffeomorphic to Y^+. This completes the proof when $c = h_2$.

In the general case when $c > h_2$, we add $c - h_2$ cancelling $1-2$ pairs and $c - h_2$ canceling $2-3$ pairs in the original handle decomposition of X given by f. After said perturbation of f, we modify the pieces accordingly. (Some modifications are required to make the pieces of the trisection diffeomorphic to each other. Other modifications are needed so that the attaching spheres of the 2–handles are embedded in the trisections srurface.) We have $X_1' = X_1 \natural^{c-h_2} S^1 \times B^3$, whose boundary is similarly decomposed as $\partial X_1' = M \cup \left(N' \#^{c-h_2} S^1 \times S^2\right)$. Additionally, we have a new generalized Heegaard splitting of $N' \#^{c-h_2} S^1 \times S^2$

$$(Y^+ \# \mathcal{H}_1) \cup (Y^- \# \mathcal{H}_2),$$

where $\mathcal{H}_1 \cup \mathcal{H}_2$ is the standard genus $c - h_2$ Heegaard splitting of $\#^{c-h_2} S^1 \times S^2$. That is, we have stabilized $Y^+ \cup Y^-$ $c - h_2$ times, once for each newly added 1–handle. The new generalized Heegaard surface F is of genus $p + h_1 - m + c - h_2$ and has b boundary components. Moreover, the original link L projects onto F as it did before. However, we now have an additional $2(c - h_2)$ link components corresponding to the newly added 2–handles. The half which correspond to the $1-2$ pair necessarily have the canceling intersection property discussed above. The half corresponding to the $2-3$ pairs project onto to F as a 0 framed unlink which bounds disks in F. Stabilizing $Y^+ \underset{F}{\cup} Y^-$ for the last time(s) near each unknot allow us to slide the links into canceling position with the new $S^1 \times S^1$ summands of F. $\square$

3.2 The Gluing Theorem

Before we state the gluing theorem, we must state a lemma regarding compression bodies.

Lemma 3. Define the quotient space $M = F_{g,b} \times I/(x,t) = (x, 1-t)$ for every $x \in \partial F_{g,b}$ and every $t \in I$. M is diffeomorphic to $F_{g,b} \times I$.

The following pictures present the proof.

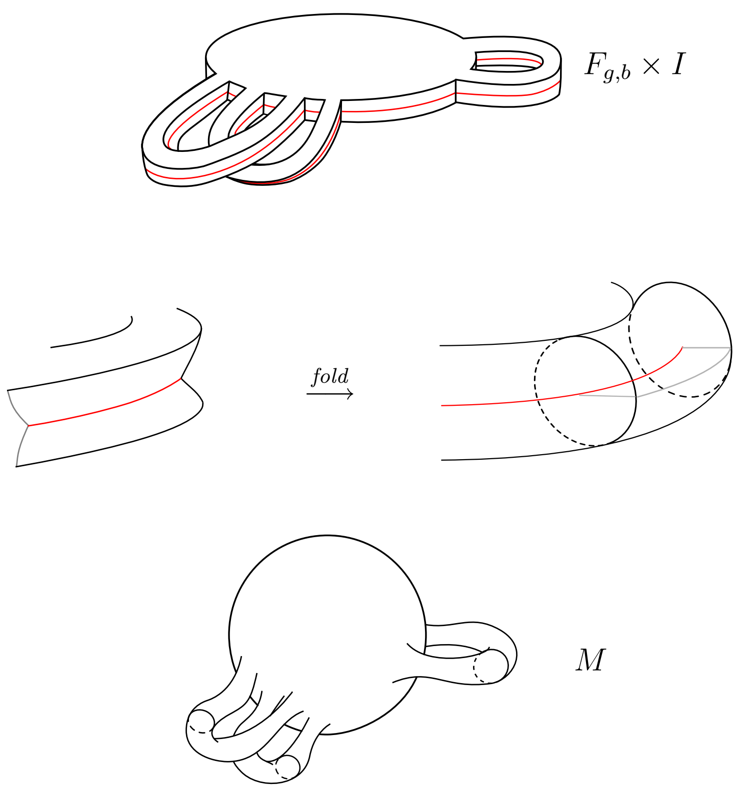

Figure 3.7: Proof of Lemma 3

Theorem 6. Let $(X, \mathcal{T}_X)$ and $(W, \mathcal{T}_W)$ be smooth, compact, connected, oriented trisected 4–manifolds with non-empty boundary. Suppose $D \subset \partial X$ is a non-empty collection of the boundary components of X and $f : D \hookrightarrow \partial W$ is an orientation reversing smooth map which respects the open books induced by $\mathcal{T}_X$ and $\mathcal{T}_W$.

i) If $D = \partial X$, then $\mathcal{T}_X$ and $\mathcal{T}_W$ induce a trisection of the closed 4-manifold $X \underset{f}{\cup} W$.

ii) If $D \neq \partial X$, then $\mathcal{T}_X$ and $\mathcal{T}_W$ induce a relative trisection on the manifold with connected boundary $X \underset{f}{\cup} W$.

Schematics for the two possible gluings are given in Figure 3.2. Note that the schematic on the right depicts the gluing of only one boundary component from each manifold, but should be thought of as "not all components get glued."

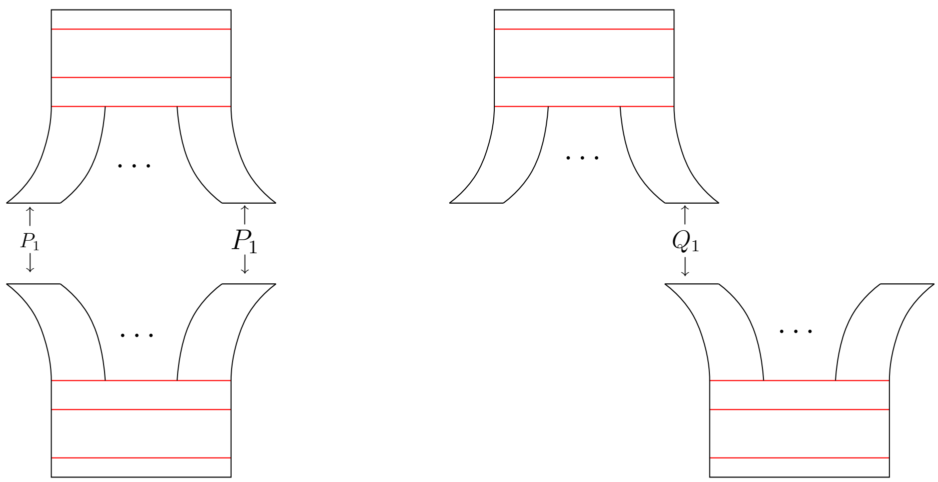

Figure 3.8: Gluing Relative Trisections

Proof. Let $P = \overset{m}{\underset{j=1}{\sqcup}} P_j$ and $Q = \overset{\mu}{\underset{j=1}{\sqcup}} Q_j$ be the pages of the open books induced by $\mathcal{T}_X$ and $\mathcal{T}_W$ respectively, where $P_j \cong F_{p_j,b_j}$ and $Q_j \cong F_{q_j,d_j}$ for each j. Additionally, let C and B denote the compression bodies which give us the X_i's and W_i's (i.e., $C \times I \cong X_i$ and $B \times I \cong W_i$). Let n and η denote the number of 3–dimensional 1–handles in the constructions of C and B respectively.

We begin with the case $D = \partial X \cong \partial W$. Our gluing is defined in the natural way, by attaching $Out(\partial X_i)$ to $Out(\partial W_i)$ via f. Our new pieces are given by

$$Z_i := X_i \cup W_i / \sim ,$$

where $f(x) = x$ for all $x \in Out(\partial X_i)$. We wish to show that Z_i is a 4–dimensional handlebody, and that $Z_i \cap Z_{i+1}$ and $Z_i \cap Z_{i-1}$ are 3–dimensional handlebodies. Since X_i and W_i are thickened compression bodies, we will reduce these to 3–dimensional arguments.

Lemma 3 tells us that $(P \times I) \underset{f}{\cup} (Q \times I) \cong \natural^l S^1 \times D^2$. Since the 1–handles in the construction of B and C are attached along the interior of level sets, the gluing and the 1–handles are independent of each other. Thus, $A := C \cup B / \sim$ is a handlebody, where we have attached $n - (m-1) + \eta - (\mu - 1)$ 1–handles to $\natural^l S^1 \times D^2$. By definition of our gluing, we have $A = In(\partial X_i) \cup \overline{In(\partial W_i)} / \sim$. Thus, $A = Z_i \cap Z_j$ is a 3–dimensional handlebody of genus $k = l + n - (m-1) + \eta - (\mu - 1)$. Noting that $Z_i = A \times I$ gives us the desired result.

The more difficult case is when we wish to result in a relative trisection. For simplicity, we will prove this case when gluing X and W along a single boundary component given by a map which takes P_1 to Q_1 as in Figure 3.2. The argument easily generalizes to multiple boundary components.

Let us view B and C as being constructed in reverse as mentioned in the previous section. The fact that the 1–handles in these constructions are attached to the interiors of P and Q allows us to glue $P_1 \times I$ to $Q_1 \times I$ before connecting components of the compression bodies. Denote $M = (P_1 \times I \underset{f}{\cup} Q_1 \times I)$. In other words, $A = C \cup B / \sim$ can be constructed by attaching 1–handles to

$$\left(\bigcup_{i=2}^{m} (P_i \times I) \right) \cup M \cup \left(\bigcup_{i=2}^{\mu} (Q_i \times I) \right) .$$

Lemma 3 again gives us $M \cong \natural^{l_i} S^1 \times D^2$ which can be constructed by attaching l_1 3–dimensional 1–handles to B^3. Thus A can be constructed as follows: Attach $m-2$ 1–handles to $P \times I$ and $\mu - 2$ 1–handles to $Q \times I$ so that each are connected. We then attach these components to B^3 (the 0-handle of M). Note that these two 1–handles giving us a connected manifold are the 1–handles which connect $P_1 \times I$ and $Q_1 \times I$ to the remaining thickened open books in $P \times I$ and $Q \times I$ respectively. They also do not increase genus.To complete the construction, we attach l_1 1–handles, coming from the construction of M. This, gives us a compression body A whose "smaller genus" end (pages of open book) is $\bigcup_{j=2}^{m} P_j \cup \bigcup_{j=2}^{\mu} Q_j$ and "larger genus" end is a surface of genus

$$\sum_{j=2}^{m} p_j + \sum_{j=2}^{\mu} q_j + (n - m + 1) + (\eta - \mu + 1) + (2p_1 + b_1 - 1) \tag{3.1}$$

with $(b - b_1) + (d - d_1)$ boundary components.

Although the new trisection genus given by (3.1) is quite involved, the idea behind the calculation is quite simple. If $\mathcal{T}_X$ and $\mathcal{T}_W$ have relative trisection surfaces F_X and F_W respictively, we obtain the new trisection surface F_Z by identifying the boundaries of F_X and F_W corresponding to the open books (P_1, ϕ_1) and (Q_1, ψ_1). $\square$

3.3 Relative Stabilizations

In this section we describe a stabilization of a relative trisections which is significantly different than that given by Gay and Kirby in that it changes the boundary data and increases the trisection genus by either one or two. This *relative stabilization* was inspired by and makes use of stabilizations of Lefschetz fibrations. Consequently, the effect a relative stabilization has on the open book decomposition of ∂X is plumbing on a Hopf band to the pages and changing the monodromy by composing with a Dehn twist along the associated vanishing cycle.

Given a relative trisection $\mathcal{T}$ of X, consider a corresponding function $f : X \to D^2$ as constructed in Theorem 5 (without identifying D^2 with $I \times I$ and projecting onto the first factor). We begin by introducing a Lefschetz singularity as in Section 2.4. In the case of multiple boundary components, a choice must be made as to where to attach the 1–handle of the canceling pair. However, we must be sure that the attaching sphere is contained in a single boundary component $M_i \subset \partial X$ with open book (P_i, ϕ_i). (Otherwise, we would be changing our 4–manifold by connecting boundary components.) We attach the 2–handle just as before, in the neighborhood of a regular value y_0, creating a singularity locally modeled by $(u, v) \mapsto u^2 + v^2$. The left half of Figure 3.10 shows a neighborhood of the singularity and a neighborhood of the vanishing cycle.

Remark 2. Notice that the attaching spheres of the 1–handle can be attached to the same binding component or to different binding components. We discuss this difference below.

Let $Z_f \subset D^2$ denote the original critical values of f before introducing the canceling pair. This is a codimension 1 set which is given by indefinite folds with finitely many cusps and crossings. We wish to show that we can "move x_0 past" all but finitely many points of Z_g. That is, choosing a different regular fiber at which to attach the singular 2–handle yields an isotopic function on X. Without loss of generality, assume $0 \notin Z_f$ and that $f^{-1}(0)$ is a fiber whose genus is maximal amongst regular fibers (i.e., $f^{-1}(0) \cong F_{g,b}$ is the trisection surface of genus g with b boundary components).

Let $\gamma : [0, 1] \to D^2$ be a smoothly embedded path from $\gamma(\varepsilon) = \vec{0}$ to $\gamma(1 - \varepsilon) = y_0$ such that:

(1) γ intersects Z_f at points $p_1, \ldots, p_n \in D^2$, none of which are cusps or crossings of Z_f,

(2) if we denote $p_i = \gamma(t_i)$, then $t_i < t_{i+1}$ for every i,

(3) the genus of the bounded fiber $f^{-1}(\gamma(t_i - \varepsilon))$ is one less than that of $f^{-1}(\gamma(t_i + \varepsilon)$.

This gives us a path as in Figure 3.9. (The conditions above are simply to ensure that γ is a path which does not intersect the same folds of Z_f more than once.) Let $M_\gamma = f^{-1}(\gamma)$, then $\gamma^{-1} \circ f|_{M_\gamma} : M_\gamma \to [0,1]$ is a Morse function such that each $f^{-1}(p_i)$ and $f^{-1}(y_0)$ are index–2 critical points. It is a standard result in Morse theory that critical points of the same index can be reordered. That is, we can modify f so that the index–2 critical point corresponding to the newly created Lefschetz singularity is attached to the fiber $f^{-1}(0)$. Finally, since it was arranged that γ missed the cusps and crossings of Z_f, we can extend the above construction to a neighborhood $N \subset D^2$ of γ, which gives a perturbed map $f : X \to D^2$ with a single Lefschetz singularity with critical value $\vec{0}$.

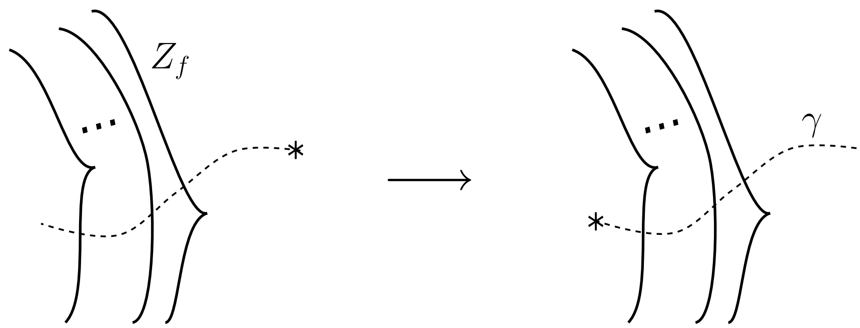

Figure 3.9: Preparing a Lefschetz Singularity for Wrinkling

Let us now perturb f in a neighborhood of $\vec{0}$ via

$$f_t(u, v) = u^2 + v^2 + tRe(u),$$

or in real coordinates

$$f_t(s, x, y, z) = (s^2 - x^2 + y^2 - z^2 + ts, 2sx_2yz).$$

For $t > 0$ the critical values of f_t are given by

$$\Gamma_t := \{(s, x, y, z) \in \mathbb{R}^4 | x^2 + s^2 + \frac{st}{2} = 0,\ y = z = 0\},$$

which defines a *triple cuspoid* pictured in Figure 3.10. In [7], Lekili shows that for $y_1, y_2 \in D^2$, where y_1 is in the interior of the triple cuspoid and y_2 is in the exterior, then the genus of $f_1^{-1}(y_1)$ is one greater than that of $f_1^{-1}(y_2)$. This perturbation is known as *wrinkling a Lefschetz singularity.* The triple cuspoid can be thought of as a Cerf graphic, where each cusp gives a canceling 1–2 pair and crossing a fold into the bounded region corresponds to attaching a 1–handle. Lekili further shows that crossing a fold into the exterior of the cuspoid corresponds to attaching a 2–handle along a curve in the the central fiber. In Figure 3.10, the colors of the attaching spheres correspond to the colors of $l_{ij} \subset D^2$ so as to indicate the isotopy class of curves determined by which fold of Γ_t each line crosses.

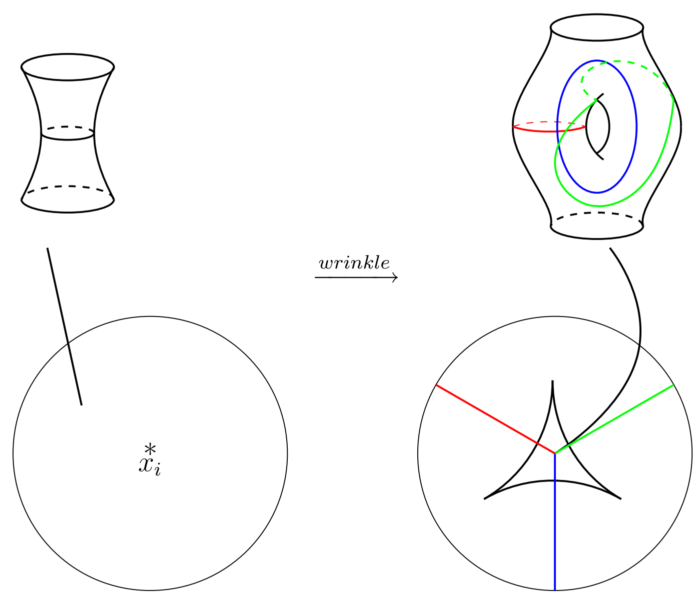

Figure 3.10: Wrinkling a Lefschetz Singularity

Notice that wrinkling is a local modification which is done on the interior of X. Thus, the action of wrinkling by itself does not modify any boundary data.

All that remains is to show that the resulting function does in fact result in a trisected 4–manifold. Let l_{ij} denote the image under the original map f of $X_i \cap X_j$ fixed as a subset of D^2. Moreover, let us arrange for the image of $X_1 \cap X_2 \cap X_3$ to be $\vec{0}$. For sufficiently small $t > 0$, we may assume that Γ_t is disjoint from Γ_f and that each l_{ij} do not intersect Γ_t at a cusp or crossing. If we now choose an identification of D^2 with $I \times I$, we can proceed to construct a trisection $\mathcal{T}'$ of X as in Theorem 5.

Definition 8. The above process is a *stabilization* of the trisection $\mathcal{T}$ relative to the open book (P_i, ϕ_i).

By construction, a relative stabilization of $\mathcal{T}$ induces a stabilization of the open book decomposition $\mathcal{T}|_M = (P_i, \phi_i)$. (Recall that this has the effect of plumbing a Hopf band onto the pages and the monodromy gets composed with a Dehn twist along the vanishing cycle.) If the feet of the 1–handle are contained in a single binding component, then the plumbing increases the number of boundary components of the page by one and fixes the genus. If different binding components are involved, then the plumbing decreases the number of boundary components of the page by one and increases the genus by one. As mentioned before, wrinkling the Lefschetz singularity increases the genus of the central fiber by one.

Let us now summarize stabilizations of $\mathcal{T}_X$ relative to (P_i, ϕ_i), resulting in a new trisection $\mathcal{T}'_X$.

- $\mathcal{T}'$ admits a decomposition of $X = X'_1 \cup X'_2 \cup X'_3$, where $X'_i \cong X_i \natural (S^1 \times B^3)$.
- If $F_{g,b}$ is the trisection surface of $\mathcal{T}$, then the trisection surface of $\mathcal{T}'$ is either $F_{g+1,b+1}$ or $F_{g+2,b-1}$.
- The induced open book $\mathcal{T}'|_M = (P'_i, \phi'_i)$ is a stabilization of (P_i, ϕ_i).

A complete uniqueness theorem for relative trisections would require a list of stabilizations which allow us to make any two trisection of a fixed 4–manifold isotopic. It is unclear as to whether or not such a list exists. However, Gay and Kirby gave the following uniqueness statement:

Theorem 7 ([3])**.** Any two relative trisections of X which induce the same open books on ∂X can be made isotopic after a finite number of interior stabilizations of both.

Now that we have relative stabilizations at our disposal, this statement can be strengthened.

Theorem 8. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are relative trisections of X such that their induced open books on ∂X can be made isotopic after Hopf stabilizations, then the two relative trisections can be made isotopic after a finite number of interior and relative stabilizations of each.

Proof. By assumption, we can perform relative stabilizations of $\mathcal{T}_1$ and $\mathcal{T}_2$ so that they induce equivalent open books on ∂X. Since relative stabilizations, in some sense, "factor through" Lefschetz singularities, we have the liberty to choose the vanishing cycles of the associated singularities, thus allowing us to ensure that the appropriate Hopf stabilizations are induced. We now call upon the uniqueness statement of Gay and Kirby to finish the proof. □

Notice that relative stabilizations behave well with gluings due to the induced Hopf stabilization on the open book. More precisely,

Lemma 4. Suppose $\mathcal{T}_Z$ and $\mathcal{T}_W$ relative trisections of Z and W with induced open books (P, ϕ) and (Q, ψ) respectively. Let $f : \bigsqcup_{r=1}^{n} M_{\phi_{i_r}} \to \bigsqcup_{r=1}^{n} M_{\psi_{j_r}}$ be an orientation reversing diffeomorphism respecting the induced open books on each boundary component, where $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\} \subset \{1, \ldots, m\}$. If $\mathcal{T}_Z'$ and $\mathcal{T}_W'$ are relative stabilizations of $\mathcal{T}_Z$ and $\mathcal{T}_W$ relative to (P_{i_1}, ϕ_{i_1}) and (Q_{j_1}, ψ_{j_1}) respectively, then f can be extended so as to induce the trisection $\mathcal{T}_Z' \cup_f \mathcal{T}_W'$ on $Z \cup_f W$.

Proof. This is quite easily done after realizing the induced stabilizations on the open books as taking the connected sum with S^3 equipped with the Hopf open book. We then extend f

via $-\mathrm{id}$, ensuring that f still respects the induced open books of $\mathcal{T}_Z'$ and $\mathcal{T}_W'$. It is essential that we perform the same type of relative stabilization to each trisection. $\square$

Remark 3. An interior stabilization is a local operation, contained in a neighborhood of a point in the trisections surface. Thus, any two interior stabilizations of a fixed trisection yield equivalent trisections. It can be shown that gluing together two copies of a single relative stabilization of the trivial trisection $\mathcal{B}$ of B^4 yields the stabilized genus three trisection of S^4. We can loosely denote this as $\mathcal{B}' \cup \overline{\mathcal{B}'} = \mathcal{S}'$. This requires *relative trisection diagrams*, defined in a forthcoming paper joint with Gay and Pinzón. Since interior stabilizations can move freely around our trisection, there is a desirable relationship between gluing and stabilizing relative trisections. Let $\mathcal{T} \natural \mathcal{B}$ denote a relative stabilization of $\mathcal{T}$ *which increases the number of boundary components* of the trisection surface. Similarly, let $\mathcal{T} \# \mathcal{S}$ denote an interior stabilization. Then the following are trisections are diffeomorphic:

i) $(\mathcal{T}_Z \cup \mathcal{T}_W) \# \mathcal{S}$

ii) $(\mathcal{T}_Z \natural \mathcal{B}) \cup (\mathcal{T}_W \natural \overline{\mathcal{B}})$

iii) $(\mathcal{T}_Z \# \mathcal{S}) \cup \mathcal{T}_W$

iv) $\mathcal{T}_Z \cup (\mathcal{T}_W \# \mathcal{S})$

The condition that our stabilization increase the number of boundary components is necessary because this is the only stabilization that can be performed on $\mathcal{B}$. However, it is reasonable to expect that performing consecutive relative stabilizations of each type to $\mathcal{B}$ is diffeomorphic to $\mathcal{B} \# \mathcal{S}$.

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